Darboux coordinates for symplectic groupoid

(1) Geometric Motivation

(2) Symplectic groupoid of unipotent uppertriangular matrices

(3) Fock - Goncharov coordinates of configuration space $\left(SL_n/B\right)^3/SL_n$.

(4) Transport matrices and maps to unipotent triangular forms

(5) Braid group action

(6) Quantum version

(1) Geometric motivation :

Σ = 2d Riemann surface. Let P be a principal G-bundle over Σ and $\mathcal{A}$ denote the space of connections on P. For a connection $A \in \mathcal{A}$, we define Atyah-Bott sympl. form

$$\omega_A : T_A \mathcal{A} \times T_A \mathcal{A} \to \mathbb{R}$$

$$\omega_A(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta \rangle$$

$T_A \mathcal{A} \simeq \Omega^1_\Sigma(\mathfrak{g})$. $\mathcal{A}$ is an affine space modelled on $\Omega^1_\Sigma(\mathfrak{g})$.

$\mathfrak{g} = Lie(G)$

$\langle \alpha \wedge \beta \rangle$ is the composition

$$\Omega^1_\Sigma(\mathfrak{g}) \times \Omega^1_\Sigma(\mathfrak{g}) \xrightarrow{\wedge} \Omega^2_\Sigma(\mathfrak{g}) \xrightarrow{\langle \cdot, \cdot \rangle} \Omega^2_\Sigma$$

where $\langle \cdot, \cdot \rangle$ is an Ad-invariant inner product on the Lie algebra $\mathfrak{g}$.

Thm (ref [Goldman]) ω is a symplectic form on $\mathcal{A}$ and the action of the group of gauge transformations $\mathcal{G}$ on $\mathcal{A}$ is Hamiltonian with respect to this symplectic structure and has moment map $\mu(A) = -F_A$

$F_A : \mathcal{A} \to \Omega^2_\Sigma(\mathfrak{g})$ is the curvature map is considered as a map

$$F : \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^*$$

via the identification $\Omega^2_\Sigma(Ad(P)) = \Omega^0_\Sigma(Ad(P))^\circ \simeq$
$\simeq \mathrm{Lie}(\mathcal{G})^*$

For a moment map we can take the Marsden-Weinstein quotient $\mathcal{A}/\!/\mathcal{G} = \mu^{-1}(0)/\mathcal{G}$.

Here $\mu^{-1}(0)$ is the finite dim space of flat connections equipped with induced symplectic structure (dually, Goldman Poisson bracket)

One can identify the space of flat connections with so-called character variety: $\mathrm{Hom}(\pi_1(\Sigma), G)/_{AdG}$

One can write the Goldman bracket using fat graphs. Fat graph parametrize hyperbolic surfaces with geodesic holes.

"Fat graph" is an embedded 3 valent graph $\Gamma \subset \Sigma$

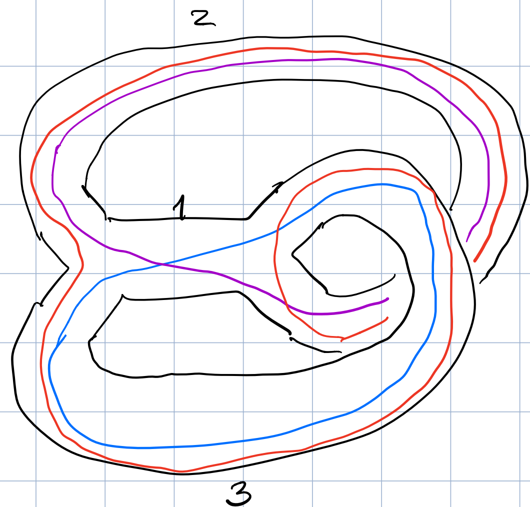
whose complement is a set of disk, each disk contains one hole.

monodromy operator along γ.

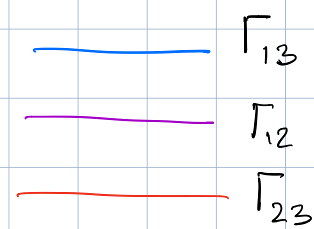
Geodesic function: loop $\gamma \in \pi_1(\Sigma) \longmapsto \mathrm{tr}(M_\gamma)$

Geometric case: $G = SL_2$. (trace does not depend on direction $\operatorname{tr} A = \operatorname{tr} A^{-1}$)

e.g.



= torus T^2 with one hole



Goldman bracket is the bracket induced on the algebra of geodesic functions.

Geodesic functions form an algebra w.r.t. multiplication and Poisson bracket.

Poisson bracket takes a form of skein relation:

$\{$  $\} = -\frac{1}{2}$  $+ \frac{1}{2}$ 

$G_1 \circ G_2$ $\quad q^{-1/2} \quad q^{1/2}$

quantum Skein

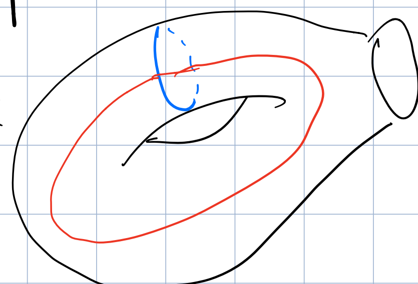
Using relation in SL_2:

$$\operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) - \operatorname{tr}A \cdot \operatorname{tr}B = 0.$$

we get $\{G_{12}, G_{23}\} = \frac{1}{2} G_{12} G_{23} - G_{13}$

+ cyclic permutations

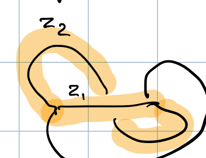
(2) Braid group action



<u>Dehn twists</u> act on the space of geodesic functions and satisfy braid group relation.

(3) Exponential shear parameters.

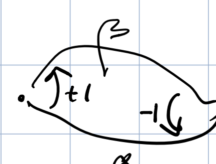
each edge of fat graph is equipped with positive parameter



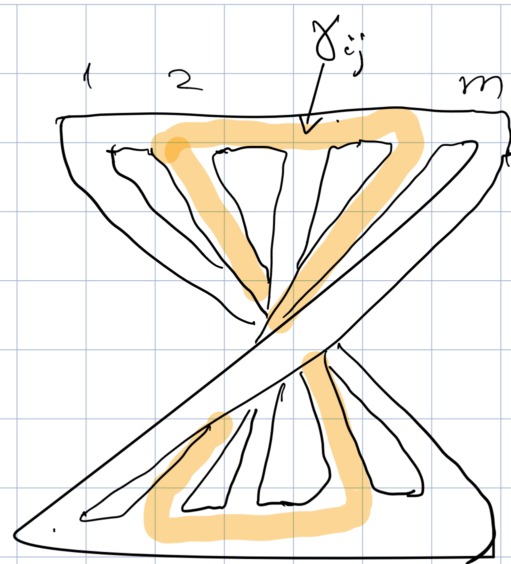
Introduce $X_i = \begin{pmatrix} 0 & -Z_1^{-\frac{1}{2}} \\ Z_1^{\frac{1}{2}} & 0 \end{pmatrix}$ $\quad L = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ $\quad R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

$G_{12} = \mathrm{tr}(L X_2 R X_1)$ etc.

<u>Statement</u>. $\{z_\alpha, z_\beta\} = \varepsilon_{\alpha\beta} z_\alpha z_\beta$ where $\varepsilon_{\alpha\beta} =$

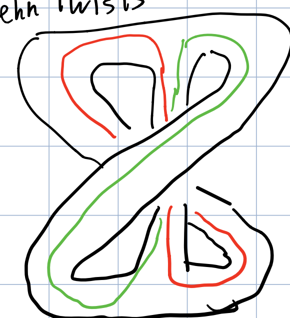


Example 2.



$a_{ij} = tr(M_{\gamma_{ij}})$

Dehn twists



Poisson bracket:

$$\{a_{ij}, a_{kl}\} = \begin{cases} 0, & j<k \\ 0, & k<i,\ j<l \\ a_{ik}a_{jl} - a_{kj}a_{il}, & i<k<j<l \\ \frac{1}{2} a_{ij}a_{jl} - a_{il} & j=k \\ a_{il} - \frac{1}{2} a_{ij}a_{il} & i=k,\ j<l \\ a_{ik} - \frac{1}{2} a_{ij}a_{kj} & j=l,\ i<k \end{cases}$$

Symplectic groupoid.

Let $\mathcal{A} \subseteq gl_n$ be a subspace of unipotent upper-triangular matrices.

$A \in \mathcal{A}$

Morphism (B, A) such that $BAB^T \in \mathcal{A}$

$\mathcal{M}$ = space of morphisms.

Standard maps.

source: $\quad s : \mathcal{M} \to \mathcal{A} \quad (B,A) \to A$

target: $\quad t : \mathcal{M} \to \mathcal{A} \quad (B,A) \to BAB^T$

injection: $\quad e : \mathcal{A} \to \mathcal{M} \quad A \to (E, A)$

inversion: $\quad i : \mathcal{M} \to \mathcal{M} \quad (B,A) \to (B^{-1}, BAB^T)$

multiplication: $\quad m : \mathcal{M}^{(2)} \to \mathcal{M} \quad ((C, BAB^T), (BAB^T)) \to (CB, A)$

$\mathcal{M}^{(2)}$ is the fibered square:

$$\begin{array}{ccc} \mathcal{M}^{(2)} & \xrightarrow{p_2} & \mathcal{M} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle s} \\ \mathcal{M} & \xrightarrow{t} & \mathcal{A} \end{array}$$

Def. Symplectic groupoid = smooth groupoid equipped with the symplectic 2-form $\omega \in \Omega^2 \mathcal{M}$ satisfying compatibility condition: $\underbrace{m^*\omega}_{\text{form on } \mathcal{M}^{(2)}} = \underbrace{p_1^*\omega + p_2^*\omega}_{\text{form on } \mathcal{M}^{(2)}}$

Poisson bracket on $\mathcal{M}$, $\pi = \omega^{-1} \in H^0(\mathcal{M}, \Lambda^2 T_{\mathcal{M}})$

$ds(\pi)$ is a bivector field P on $\mathcal{A}$

(it does not depend on the point in the fiber $s^{-1}(A)$)

This gives a Poisson structure on $\mathcal{A}$.

(Nelson-Regge, Gavrilik-Klimyk, Bondal, Dubrovin, Ugaglia, Boalch)

- Casimirs: coefficients of characteristic polynomial $\chi_A(\lambda) = \det(A + \lambda A^T)$ generate algebra of Casimirs

Geometric situation (Example 2) has dimension growing linearly with n.

$\dim \mathcal{A}$ grows $\sim n^2/2$

Example 2 describes a symplectic leaf in $\mathcal{A}$.

Problems: (1) describe Darboux coordinates of a general symplectic leaf.

(2) braid group action: generators $\beta_{i,i+1}[A] = B_{i,i+1} A B_{i,i+1}^T$

$$B_{i,i+1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & 0 \\ & & 1 & & & \\ & & & a_{i,i+1} & -1 & \\ & 0 & & 1 & 0 & \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix}$$

Find cluster representation of braid group action.

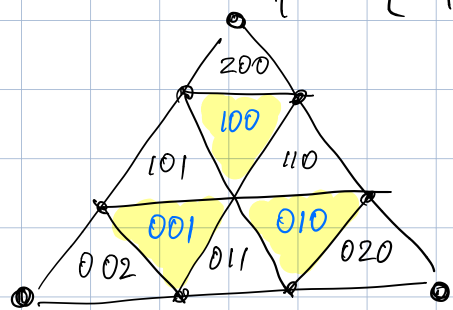
(3) quantum version.

Framed moduli space $\mathcal{X}_{SL_n,\Delta}$

$$\simeq \left(SL_n/B\right)^3 / Ad_{SL_n}$$

Fock - Goncharov parameters (ex. SL_3)

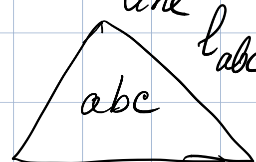
$$F_1^\circ = \{ F_1^1 \subset F_1^2 \subset \mathbb{C}^3 \}$$



$F_2^\circ = \ldots$

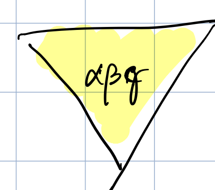
$F_3^\circ = \ldots$

line



$$l_{abc} = F_1^{n-a} \cap F_2^{n-b} \cap F_3^{n-c}$$

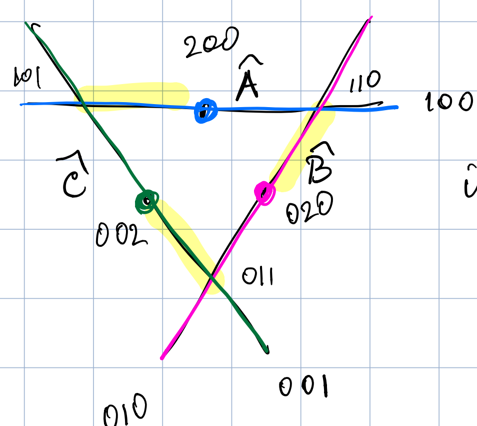
note $a+b+c = n-1$



plane

$$P_{abc} = F_1^{n-a} \cap F_2^{n-b} \cap F_3^{n-c}$$

$$a+b+c = n-2$$



3 space

inner vertex.

$$\longrightarrow V_{abc} = F_1^{n-a} \cap F_2^{n-b} \cap F_3^{n-c}$$

$$a+b+c = n-3$$

contains $P_{a+1\,bc},\ P_{a,b+1,c},\ P_{a,b,c+1}$

and hence 6 lines

$$\ell_{a+2,b,c},\ \ell_{a+1,b+1,c},\ \ell_{a+1,b,c+1},$$
$$\ell_{a,b+2,c},\ \ell_{a,b+1,c+1},\ \ell_{a,b,c+2}$$

we get 3 flags $\hat{A}, \hat{B}, \hat{C}$ in $\hat{V}_{abc}$

Let $\hat{A} = (\bar{a}_1, \bar{a}_1 \wedge \bar{a}_2)$, $\hat{B} = (\bar{b}_1, \bar{b}_1 \wedge \bar{b}_2)$

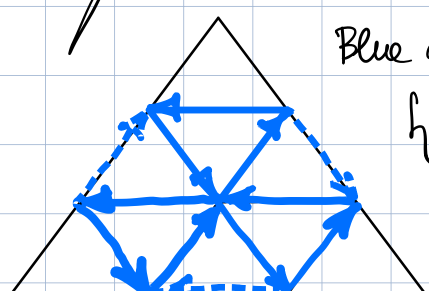
$\hat{C} = (\bar{c}_1, \bar{c}_1 \wedge \bar{c}_2)$ $\quad \bar{a}_i, \bar{b}_i, \bar{c}_i \in V_{abc}$

we assign to inner vertex abc, $a+b+c=n-3$

parameter $$x_{abc} = \frac{\langle a_1\wedge a_2 \wedge b_1\rangle \langle b_1 \wedge b_2 \wedge c_1\rangle \langle c_1 \wedge c_2 \wedge a_1\rangle}{\langle a_1\wedge a_2 \wedge c_1\rangle \langle b_1 \wedge b_2 \wedge a_1\rangle \langle c_1 \wedge c_2 \wedge b_1\rangle}$$

projective invariant.

— Lattice points on sides are also assigned some parameters.

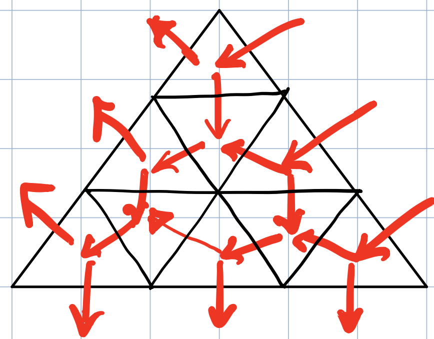


Blue quiver: Poisson bracket

$$\{X_v, X_w\} = \varepsilon_{vw} X_v X_w$$

$$\varepsilon_{vw} = \#(v \to w) - \#(v \leftarrow w) + \tfrac{1}{2}(v \dashrightarrow w) - \tfrac{1}{2}(v \dashleftarrow w)$$

Transport matrix



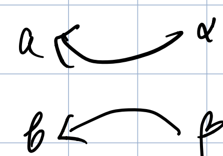
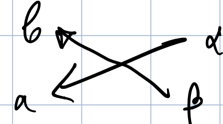
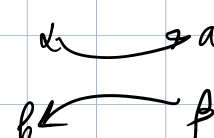
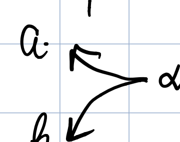
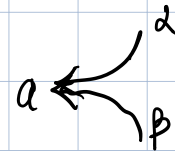
weight of path p

$$W_p = \prod_{\substack{\text{vertex } v \\ \text{on the right} \\ \text{of } p}} \text{weight}(v)$$

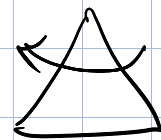
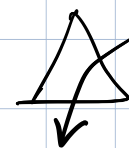
Source i, sink j $\rightsquigarrow$

transport element $$\mathcal{M}_{ij} = \sum_{\substack{\text{oriented path} \\ p: i \rightsquigarrow j}} w(P_{i \rightsquigarrow j})$$

Th (GSV'08, ChS'20 quantum)

- $\{\mathcal{M}_{a\alpha}, \mathcal{M}_{b\beta}\} = 2\,\mathcal{M}_{a\alpha}\mathcal{M}_{b\beta}$ 
- $\{\mathcal{M}_{a\alpha}, \mathcal{M}_{b\beta}\} = 0$  or 
- $\{\mathcal{M}_{a\alpha}, \mathcal{M}_{b\alpha}\} = \mathcal{M}_{a\alpha}\mathcal{M}_{b\alpha}$ 
- $\{\mathcal{M}_{a\alpha}, \mathcal{M}_{a\beta}\} = \mathcal{M}_{a\alpha}\mathcal{M}_{a\beta}$ 

Transport matrices

$\mathcal{M}_1$  $\qquad\qquad \mathcal{M}_2$ 

$$M_1 = S\,\mathcal{M}_1 \qquad\qquad M_2 = S\,\mathcal{M}_2$$

where $S = \begin{bmatrix} \emptyset & & 1^{-1^{\cdot^{\cdot^{\cdot}}}} \\ & 1^{-1} & \\ 1^{-1} & & \emptyset \end{bmatrix}$ $\qquad M_1 = \left({}^{*}\!/\!{}_{0}\right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad M_2 = \left({}^{0}\!/\!{}_{*}\right)$

Cor. Poisson bracket between elements of $\mathcal{M}_i$ is the standard Poisson-Lie Sklyanin bracket.

$\{M_i \overset{\otimes}{,} M_j\}$ is Goldman Poisson bracket

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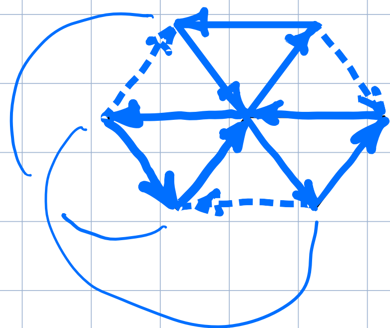
Main theorem (Chekhov - S. '20)

$A = M_1^T M_2$ satisfies Bondal Poisson bracket on $\mathcal{A}$

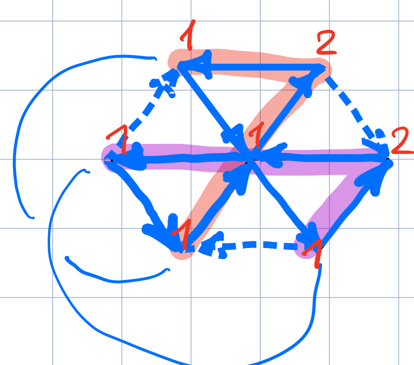
$A = \left({}_{0}\backslash^{*}\right)$. Fock-Goncharov parameters Z_{abc} give Darboux type coordinates for the Poisson bracket.

Quiver of Poisson bracket.



gluing vertices (multiplying corresponding weights)

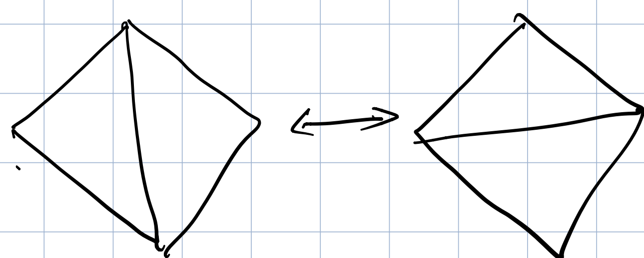
$(n-1)$ Casimirs :



Choosing values of these Casimirs we make $A = M_1^T M_2$ unipotent.

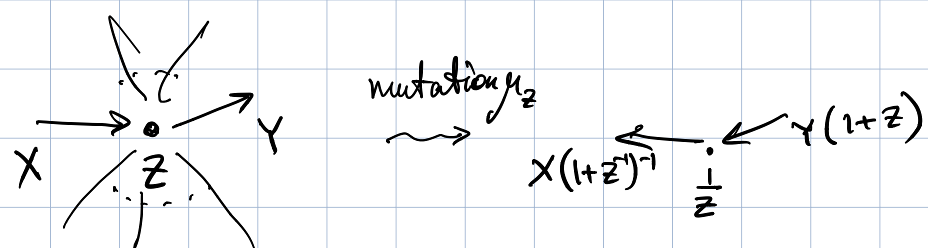
Cluster transformations.

- Flip of triangulations leads to

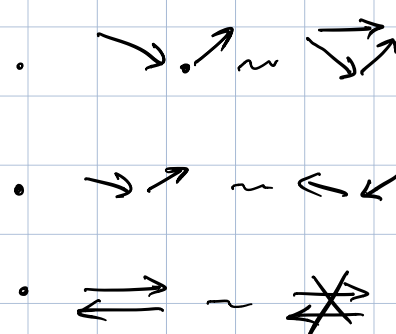


Fock - Goncharov coordinates change

- One can realize flips in a sequence of cluster mutations

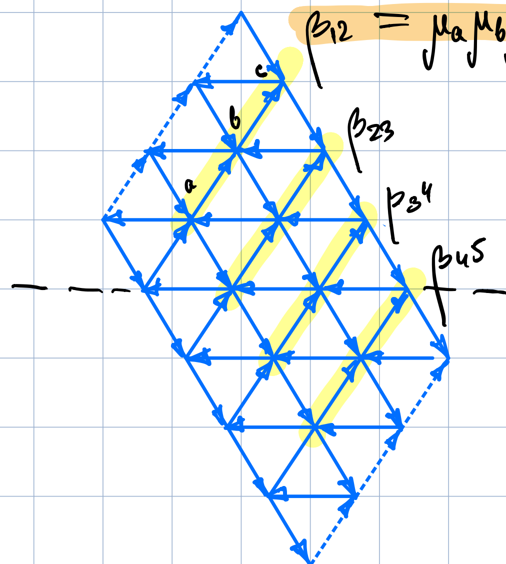


Quiver mutation in 3 steps



Braid group action on $\mathcal{A}$

$\beta_{12} = \mu_a \mu_b \mu_c \mu_b \mu_a$



Statement: $\beta_{i,i+1}$ are generators of braid group action.

Quantum version (Chekhov – S. '20)

$$A^{h} = \begin{bmatrix} q^{-1/2} & a^{h}_{12} & & a^{h}_{1n} \\ & \ddots & & \\ & & \ddots & a^{h}_{n-1\,n} \\ 0 & & & q^{-1/2} \end{bmatrix}$$

satisfy the reflection equation:

$$(*) \quad R_{12}(q)\, \overset{1}{A}{}^{\hbar}\, R^{t_1}_{12}(q)\, A^{\hbar} = \overset{2}{A}\, R^{t_1}_{12}(q)\, \overset{1}{A}{}^{\hbar}\, R_{12}(q)$$

$$R_{12}(q) = q^{-1/k}\Big[\sum_{i,j} \overset{1}{e}_{ii} \otimes \overset{2}{e}_{jj} + \sum_{i}(q-1)\, \overset{1}{e}_{ii} \otimes \overset{2}{e}_{ii} + \sum_{j>i}(q-q^{-1})\, \overset{1}{e}_{ij} \otimes \overset{2}{e}_{ji}\Big]$$

<u>Thm</u>. (Chekhov-S, Mazzocco-Rubtsov-Chekhov, A.Shapiro-Schrader)

$$\overset{1}{M}{}_1^h \otimes \overset{2}{M}{}_2^h = \overset{2}{M}{}_2^h \otimes \overset{1}{M}{}_1^h\, R_{12}(q)$$

<u>Cor</u> $*$ holds for $A^h = (M_1^h)^T M_2^h$